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PRESSURE POTENTIAL FORMULATION OF 2-D STOKES PROBLEMS IN MULTIPLY-CONNECTED DOMAINS

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SUMMARY

In general Stokes problems, no boundary conditions exist for the pressure. But pressure is an $L^2(\Omega)$ function and can uniquely be represented as the divergence of a precisely defined vector field. In the 2-D case, this vector field can in turn be represented as the sum of a gradient (of a pressure-potential) and the curl of a second scalar potential. The latter potential is entirely determined by the first one. A variational equation is obtained for such pressure potential class, which exists and is uniquely characterized. This variational problem is well-posed. Finite element approximations can easily be realized and ensure high convergence rates for the $L^2(\Omega)$ norm of the pressure.

KEY WORDS Stokes Problems Pressure Pressure Potential Incompressible

1. INTRODUCTION

The determination of the pressure in a Stokes problem has long since been known to be a difficult problem. Major progress towards the characterization of the pressure was realized by Girault and Raviart,[1] by presenting a *simultaneous* characterization for the *velocity* and *pressure*, via the *mixed variational principle*. Recently, some papers have been devoted to the question of *separate* determination of the *pressure*.[2,3] These results are, however, restricted to the particular case $\mathbf{f}\in H(div;\Omega)$. We obtain in this paper a general approach, which solves the question of pressure characterization. The starting point of our method is a weak formulation of the operator equation. A thorough understanding of the underlying mechanism, together with representation theorems for scalar functions in $L_0^2(\Omega)\doteq\{q\in L^2(\Omega)|\int_\Omega q\,dx=0\}$ (a handy substitute for $L^2(\Omega)/\mathbb{R}$), enables us to solve this problem of the separate or sequential characterization of the pressure in Stokes problems (section 2). More particularly, in the 2-dimensional case we go deeper into the heart of this problem by formulating characterizations for potentials that uniquely represent the pressure. These results are stated in variational form, and the well-posedness is proved (section 4). Finally we show that our method is well suited for constructive purposes and adapts easily to the finite element approximation (section 5).

The Stokes problem is concerned with the *stationary flow*, of an *incompressible viscous fluid*, in some domain Ω of $\mathbb{R}^n$. This motion, described both by a *velocity field* $\mathbf{u}$ and a *pressure function* P, results from the action of a *force* $\mathbf{f}$ (representing a density of volumic forces) in the presence of a prescribed *velocity pattern* $\mathbf{g}$ on the whole of the boundary of Ω (denoted Γ). We consider thus a domain Ω, which will be an *open bounded multiply-connected subset* of $\mathbb{R}^n$, whose boundary

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$$\Gamma = \bigcup_{i=0}^{N} \Gamma_i,$$

is *Lipschitz-continuous* (Reference 4, pp. 14–15), and composed of $N+1$ connected components Γ_i, where we consider Γ_0 to be the outer boundary of the domain Ω. As is well known, for every $\mathbf{f} \in [H^{-1}(\Omega)]^n$ and every $\mathbf{g} \in [H^{+1/2}(\Gamma)]^n$ satisfying

$$\int_\Gamma \mathbf{g} \cdot \mathbf{v} \, d\gamma = 0,$$

there exists a unique pair $(\mathbf{u}, P) \in [H^1(\Omega)]^n \times L_0^2(\Omega)$:

$$\begin{aligned} -\eta \, \Delta \mathbf{u} + \mathbf{grad}\, P &= \mathbf{f}, && \text{in } \Omega, \\ \mathit{div}\, \mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}, && \text{on } \Gamma, \end{aligned} \tag{1}$$

η being the *dynamic viscosity* and $\mathbf{v}$ the outer normal.

2. BASIC CHARACTERIZATIONS OF THE PRESSURE ($\mathbb{R}^n$)

From the operator formulation of the Stokes problem, we deduce by using the duality form $\langle \cdot , \cdot \rangle_{[H_0^1(\Omega)]^n}$, that for all $\mathbf{w} \in [H_0^1(\Omega)]^n$, we have

$$\langle \mathbf{f}, \mathbf{w} \rangle_{[H_0^1(\Omega)]^n} = \langle \mathbf{grad}\, P, \mathbf{w} \rangle_{[H_0^1(\Omega)]^n} + a(\mathbf{u}, \mathbf{w}), \tag{2}$$

where for $\mathbf{v}$, $\mathbf{w} \in [H^1(\Omega)]^n$ we denote:

$$a(\mathbf{v}, \mathbf{w}) \doteq \eta \left[(\mathbf{grad}\, \mathbf{v}, \mathbf{grad}\, \mathbf{w})_{0,\Omega} + \sum_{ij} (\partial_i v_j, \partial_j w_i)_{0,\Omega} \right].$$

As is well known (Reference 1, p. 53), the velocity field $\mathbf{u} \in [H^1(\Omega)]^n$: $\mathit{div}\, \mathbf{u} = 0$ *in* Ω, $\mathbf{u} = \mathbf{g}$ *on* Γ can be characterized independently from the pressure and satisfies:

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle_{[H_0^1(\Omega)]^n}, \quad \forall \mathbf{v} \in V, \tag{3}$$

where $V = \{ \mathbf{v} \in [H_0^1(\Omega)]^n \mid \mathit{div}\, \mathbf{v} = 0 \ \text{in} \ \Omega \}$. Moreover, since the Hilbert space $([H_0^1(\Omega)]^n, |\cdot|_{1,\Omega})$ admits a direct sum decomposition: $[H_0^1(\Omega)]^n = V \oplus V^\dagger$, where $V^\dagger$ is the orthogonal complement of V in $([H_0^1(\Omega)]^n, |\cdot|_{1,\Omega})$, we obtain the following basic result.

Theorem 1. *The pressure P, defined in the Stokes problem (1), is uniquely characterized by the problem:*

$$? \ P \in L_0^2(\Omega) : (P, \mathit{div}\, \mathbf{w})_{0,\Omega} = \langle \mathbf{L}, \mathbf{w} \rangle, \quad \forall \mathbf{w} \in V^\dagger, \tag{4}$$

where

$$\langle \mathbf{L}, \mathbf{w} \rangle = a(\mathbf{u}, \mathbf{w}) - \langle \mathbf{f}, \mathbf{w} \rangle_{[H_0^1(\Omega)]^n} \tag{5}$$

Proof: We introduce first the operator $B^* \in L(L_0^2(\Omega), [H^{-1}(\Omega)]^n)$, adjoint operator of $B \in L([H_0^1(\Omega)]^n, [L_0^2(\Omega)]^*)$ defined as

$$-(q, \mathit{div}\, \mathbf{v})_{0,\Omega} = \langle B^* q, \mathbf{v} \rangle_{[H_0^1(\Omega)]^n} = \langle B\mathbf{v}, q \rangle_{L_0^2(\Omega)},$$

and the operator $A \in L([H_0^1(\Omega)]^n, [H^{-1}(\Omega)]^n)$ associated to the bilinear form $a(\cdot,\cdot)$ as $a(\mathbf{v}, \mathbf{w}) = \langle A\mathbf{v}, \mathbf{w} \rangle_{[H_0^1(\Omega)]^n}$. We deduce from (2) that $P = [B^*]^{-1}(\mathbf{f} - A\mathbf{u})$. Now B^* is an isomorphism from $L_0^2(\Omega)$ onto V° (Reference 1, p. 41), where

$$V^\circ = \{\mathbf{w}\in[H^{-1}(\Omega)]^n \mid \langle \mathbf{w}, \mathbf{u}\rangle_{[H_0^1(\Omega)]^n} = 0, \quad \forall \mathbf{v}\in V\},$$

is the *polar* set of V. By the characterization of the velocity field $\mathbf{u}$ we have that (3) holds and thus $(\mathbf{f} - A\mathbf{u})\in V^\circ$. We obtain the desired result. ■

A second characterization can further be obtained by taking into account (Reference 1, p. 33) that the mapping $div: V^\dagger \to L_0^2(\Omega)$ is an *isomorphism*, element of $L((V^\dagger, |\cdot|_{1,\Omega}), (L_0^2(\Omega), \|\cdot\|_{0,\Omega}))$. The following property plays a role in the proof of the second characterization.

Proposition 1. *On the linear space* $V^\dagger$, $\|div\cdot\|_{0,\Omega}$ *is a norm equivalent to* $|\cdot|_{1,\Omega}$ *and* $\|\cdot\|_{1,\Omega}$.

Proof. Since *div* is an isomorphism on $V^\dagger$, we have for every $\mathbf{v}\in V^\dagger$ that

$$\frac{1}{\|div^{-1}\|}.|\mathbf{v}|_{1,\Omega} \leqslant \|div\,\mathbf{v}\|_{0,\Omega} \leqslant \|div\|.|\mathbf{v}|_{1,\Omega},$$

which proves the assertion. ■

We are now ready to formulate the second characterization for the pressure solution of a Stokes problem.

Theorem 2. *The pressure P, defined in the Stokes problem (1), is given as* $P = div\,\mathbf{z}$ *where the vector field* $\mathbf{z}$ *is uniquely characterized by the problem:*

$$?\ \mathbf{z}\in V^\dagger : (div\,\mathbf{z}, div\,\mathbf{w})_{0,\Omega} = \langle \mathbf{L}, \mathbf{w}\rangle, \quad \forall\, \mathbf{w}\in V^\dagger. \tag{6}$$

with $\langle \mathbf{L}, \cdot\rangle$ *defined by* (5).

Proof: The existence and unicity of the vector $\mathbf{z}$ solution of (6) follows from the *Lax–Milgram theorem* on $(V^\dagger, |\cdot|_{1,\Omega})$. Indeed $(div., div.)_{0,\Omega}$ is a bilinear continuous mapping on $V^\dagger$, as this follows from proposition 1 and the Schwarz inequality. The ellipticity of this bilinear form on $(V^\dagger, \|div.\|_{0,\Omega})$ is a consequence of its definition, and in virtue of proposition 1 is also verified on $(V^\dagger, |\cdot|_{1,\Omega})$. Finally that $\langle \mathbf{L},.\rangle$ is a linear continuous function follows from the definition of $\mathbf{f}$ as an element of $[H^{-1}(\Omega)]^n$. The partial mapping $a(\mathbf{u},.)$ is in turn also linear and continuous on $(V^\dagger, |\cdot|_{1,\Omega})$ since it is linear and continuous on $([H_0^1(\Omega)]^n, |\cdot|_{1,\Omega})$ (Reference 1, p. 52). ■

Theoretically speaking, theorems 1 and 2 characterize completely the pressure function P and the velocity field $\mathbf{z}\in V^\dagger : P = div\,\mathbf{z}$. However, the main drawback of (4) and (6) is the space $V^\dagger$ which offers fewer possibilities to be realized, e.g. by using finite element approximations. In the sequel, the main results originate from the representation of vector fields by potentials and stream functions, and therefore we restrict attention to the 2-D case. Obviously, it should be possible to obtain similar results in the 3-D case, but this requires more elaborate developments concerning the potentials involved. From now on, the domain Ω will be an *open bounded multiply connected subset of* $\mathbb{R}^2$, whose *boundary is Lipschitz-continuous.* In addition, and to ensure the existence of regular potentials and stream functions, as required in future developments, we suppose Γ to be a C^2*-manifold of dimension 1.*

3. REPRESENTATION OF VECTOR FIELDS IN $V^\dagger$ ($n = 2$)

In the sequel, we are particularly interested in the representation of vector fields $\mathbf{v}$, belonging to

$$V^\dagger = \{\mathbf{v}\in[H_0^1(\Omega)]^2 \mid (\mathbf{grad}\,\mathbf{v}, \mathbf{grad}\,\mathbf{w})_{0,\Omega} = 0, \quad \forall\, \mathbf{w}\in V\}. \tag{7}$$

We deduce first an equivalent characterization.

Proposition 2. *The linear space $V^\dagger$ defined in (7) can equivalently be stated as*

$$V^\dagger = \{\mathbf{v}\in[H_0^1(\Omega)]^2 \mid (curl\,\mathbf{v}, curl\,\mathbf{w})_{0,\Omega} = 0, \quad \forall\, \mathbf{w}\in V\}.$$

Proof. For any $\mathbf{v}\in[H^1(\Omega)]^2$ and for any $\mathbf{w}\in[H_0^1(\Omega)]^2$ we have that

$$(\mathbf{grad}\,\mathbf{v}, \mathbf{grad}\,\mathbf{w})_{0,\Omega} = (curl\,\mathbf{v}, curl\,\mathbf{w})_{0,\Omega} + (div\,\mathbf{v}, div\,\mathbf{w})_{0,\Omega}. \qquad \blacksquare$$

At this stage, what is important to note about the function $curl\,\mathbf{v}$ for some $\mathbf{v}\in V^\dagger$, is that $curl\,\mathbf{v}\in L_0^2(\Omega)$ and $\Delta\, curl\,\mathbf{v} = 0$ *in* Ω.

Taking now into account proposition 2, we immediately deduce a first representation theorem for elements in $V^\dagger$.

Theorem 3. *For every $\mathbf{v}\in V^\dagger$, there exists a unique potential class $\dot\varphi\in H^1(\Omega)/\mathbb{R}$ and a uniquely defined stream function class $\dot\psi\in H^1(\Omega)/\mathbb{R}$ such that:*

$$\mathbf{v} = \mathbf{grad}\,\varphi + \mathbf{curl}\,\psi \ \text{ in } [H_0^1(\Omega)]^2 \text{ and } \gamma_\nu(\mathbf{grad}\,\varphi) = 0 \text{ in } H^{+1/2}(\Gamma).$$

The equivalence classes $\dot\varphi \subset H^2(\Omega)$ and the functional relation M such that $\dot\psi = M(\dot\varphi) \subset H^2(\Omega)$ are defined as follows:

1. $\dot\varphi\in H^1(\Omega)/\mathbb{R}$:

$$(\mathbf{grad}\,\varphi, \mathbf{grad}\,\chi)_{0,\Omega} = -(div\,\mathbf{v}, \chi)_{0,\Omega}, \quad \forall\,\chi\in H^1(\Omega),$$

2. $M\colon \dot\varphi\in H^1(\Omega)/\mathbb{R} \to \dot\psi\in H^1(\Omega)/\mathbb{R}$, *such that* $\psi\in\dot\psi$ *satisfies:*

$$\psi\in H^2(\Omega)\colon \partial_\tau\psi = 0 \text{ on } \Gamma, \quad \partial_\nu\psi = \gamma_\tau(\mathbf{grad}\,\varphi) \text{ on } \Gamma,$$

$$(\Delta\psi, \Delta\chi)_{0,\Omega} = 0, \quad \forall\,\chi\in\Psi,$$

where $\Psi \doteq H_0^2(\Omega)\oplus span\,\{\psi_i \mid i = 1,\ldots,N\}$, *with* ψ_i; $i = 1,\ldots,N$ *defined by*

$$\psi_i\in H^2(\Omega)\colon \psi_i = \delta_{ij} \text{ on } \Gamma_j\colon j = 0,\ldots,N, \quad \partial_\nu\psi_i = 0 \text{ on } \Gamma,$$

$$(\Delta\psi_i, \Delta\chi)_{0,\Omega} = 0, \quad \forall\,\chi\in H_0^2(\Omega). \tag{8}$$

Proof. Let $\mathbf{v}$ be a fixed, but arbitrary element in $V^\dagger \subset [H_0^1(\Omega)]^2$. Let us then consider the following Neumann problem:

$$?\ \dot\varphi\in H^1(\Omega)/\mathbb{R}\colon -\Delta\varphi = -div\,\mathbf{v} \text{ in } \Omega, \quad \partial_\nu\varphi = 0 \text{ on } \Gamma.$$

Such a $\dot\varphi$ always exists, is uniquely defined by this problem and moreover any $\varphi\in\dot\varphi$ is also in $H^2(\Omega)$. To verify the last assertion, we note that since $\mathbf{v}\in[H_0^1(\Omega)]^2$, we have that $div\,\mathbf{v}\in L_0^2(\Omega)$ and take into account that by hypothesis on Ω, Γ is a C^2-manifold of dimension 1 (Reference 5, p. 68–73). At this stage, what is important to verify is that by this construction, the vector $\mathbf{w} \doteq \mathbf{v} - \mathbf{grad}\,\varphi$ is an element of the space $[H^1(\Omega)]^2 \cap H_0(div;\Omega)$. Such a vector field $\mathbf{w}$ can uniquely by represented by a stream function class $\dot\psi$ such that: $\mathbf{w} = \mathbf{curl}\,\psi$ in $[H^1(\Omega)]^2$. To this end, we first build some function ψ^* satisfying: $\psi^*\in \mathrm{H}^2(\Omega)$, $\partial_\nu\psi^* = -\gamma_\tau\mathbf{w}$ in $H^{+1/2}(\Gamma)$ and $\partial_\tau\psi^* = 0$ *in* $H^{+1/2}(\Gamma)$. *There always exist some* $H\in H^{+3/2}(\Gamma)$ *such that* $DH = 0$ *(i.e.* piecewise constants*)*. Taking now into account Reference 1, p. 5, Th. 1.5, there always exists some ψ^* in $H^2(\Omega)$ that satisfies $\gamma_0\psi^* = H$ and $\gamma_1\psi^* = -\gamma_\tau\mathbf{w}$. Next we consider the vector field $\mathbf{w}_0 \doteq \mathbf{w} - \mathbf{curl}\,\psi^*$. What is important to observe now is that $\mathbf{w}_0$ is in V. According to Reference 6, §6, there exists a unique stream function $\psi^0\in\Psi$, such that $\mathbf{w}_0 \doteq \mathbf{curl}\,\psi^0$ is in V while $\psi^0\in\Psi$ is characterized by the following variational equation:

$$(\Delta\psi^0, \Delta\chi)_{0,\Omega} = -(curl\,\mathbf{w}_0, \Delta\chi)_{0,\Omega}, \quad \forall \chi \in \Psi. \tag{9}$$

From (9) we deduce that $(curl(\mathbf{w}_0 - \mathbf{curl}\,\psi^0), curl\,\mathbf{v})_{0,\Omega} = 0$ for all $\mathbf{v} \in V$ and we obtain that $|\mathbf{w}_0 - \mathbf{curl}\,\psi^0|_{1,\Omega} = 0$. Since on $H_0^1(\Omega)$ the norms $|\cdot|_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ are equivalent, we finally deduce that $\|\mathbf{v} - \mathbf{grad}\,\varphi - \mathbf{curl}(\psi^0 + \psi^*)\|_{1,\Omega} = 0$. We can thus conclude that the identification $\mathbf{v} = \mathbf{grad}\,\varphi + \mathbf{curl}(\psi^0 + \psi^*)$ holds in $[H^1(\Omega)]^2$. ■

It is important to stress here that in the representation of $\mathbf{v}$ as $\mathbf{v} = \mathbf{grad}\,\varphi + \mathbf{curl}\,\psi$, φ depends only on $div\,\mathbf{v}$ while ψ is a function of φ only and does *not* require any information on $curl\mathbf{v}$, even though only $\Delta\, curl\,\mathbf{v} = 0$ is *in* Ω.

Let us introduce the following subset of $H^1(\Omega)/\mathbb{R}$:

$$\dot{\Theta}_1 = \{\dot{\varphi} \in H^1(\Omega)/\mathbb{R} \mid \dot{\varphi} \subset H^2(\Omega), \quad \partial_\nu \varphi = 0 \text{ on } \Gamma : \varphi \in \dot{\varphi}\}. \tag{10}$$

This set of equivalence classes is easily verified to be a linear subspace of the quotient space $H^1(\Omega)/\mathbb{R}$.

Proposition 3. *The linear space $\dot{\Theta}_1$, defined in (10), is a closed subspace of $H^1(\Omega)/\mathbb{R}$. The application:*

$$n : \dot{\varphi} \in \dot{\Theta}_1 \rightarrow \|\Delta\varphi\|_{0,\Omega}, \quad \varphi \in \dot{\varphi},$$

is a norm on $\dot{\Theta}_1$.

Proof: 1. Consider a sequence $(\dot{\varphi}_l)$ of elements in the space $\dot{\Theta}_1$, which converges to the class $\dot{\varphi} \in H^1(\Omega)/\mathbb{R}$, where $\dot{\varphi} \subset H^2(\Omega)$. Since for every $\varphi_k \in \dot{\varphi}_k$, we have that $\gamma_1 \varphi_k$ converges to $\gamma_1 \varphi$ in $H^{+1/2}(\Gamma)$ for any $\varphi \in \dot{\varphi}$, it contains a subsequence, which converges almost everywhere to $\gamma_1 \varphi$ and thus $\gamma_1 \varphi = 0$ on Γ, i.e. $\partial_\nu \varphi = 0$ on Γ. This implies that $\dot{\varphi}$ belongs to $\dot{\Theta}_1$. 2. That $\|\Delta\varphi\|_{0,\Omega}$ is a norm over the space $\dot{\Theta}_1$ can be verified as follows. Consider an element $\dot{\varphi} \in \dot{\Theta}_1$ for which $n(\dot{\varphi}) = 0$. This means that $\|\Delta\varphi\|_{0,\Omega} = 0$ and thus $\Delta\varphi = 0$ in Ω, for any $\varphi \in \dot{\varphi}$. Consequently $\dot{\varphi}$ is a solution of the Neumann problem: $\Delta\varphi = 0$ in Ω and $\partial_\nu \varphi = 0$ on Γ. Obviously $\varphi \in \dot{\varphi}$ is a constant in Ω and thus $\dot{\varphi} = 0$ in $H^1(\Omega)/\mathbb{R}$. ■

We further consider the following set in $H^1(\Omega)/\mathbb{R}$:

$$\dot{\Theta}_2 = \left\{ \dot{\varphi} \in H^1(\Omega)/\mathbb{R} \mid \dot{\varphi} \subset H^2(\Omega), \quad \partial_\tau \varphi = 0 \text{ on } \Gamma, \right.$$
$$\left. \int_{\Gamma_i} \partial_\nu \varphi \, d\gamma = 0 : i = 0, 1, \ldots, N, \quad \varphi \in \dot{\varphi} \right\}. \tag{11}$$

It is important for our analysis to mention the following constructive property for the equivalence classes, elements of this set $\dot{\Theta}_2$.

Proposition 4. *For every $\dot{\varphi} \in \dot{\Theta}_2$, there exist uniquely defined constants $(d_1, \ldots, d_N) \in \mathbb{R}^N$ such that for $\varphi \in \dot{\varphi}$, we have that:*

$$\gamma_0 \varphi = c \text{ on } \Gamma_0, \quad \gamma_0 \varphi = c + d_i \delta_{ij} \text{ on } \Gamma_j, \quad j = 1, \ldots, N,$$

for some $c \in \mathbb{R}$.

Proof: Let us consider some $\dot{\varphi}$ in $\dot{\Theta}_2$. Since for any $\varphi \in \dot{\varphi}$ we have $\partial_\tau \varphi = 0$ on Γ and thus $\gamma_0 \varphi$ is constant on every connected component Γ_j of the boundary (Reference 7, p. 153 a.f.).

It is easily verified that every $\varphi\in\dot{\varphi}$ can be represented as:

$$\varphi = c + \varphi_0 + \sum_{i=1}^{N} d_i\varphi_i, \tag{12}$$

where (i) $c\in\mathbb{R}$, (ii) $\varphi_0\in H_0^1(\Omega): -\Delta\varphi_0 = -\Delta\varphi$ in Ω, and (iii) for every $i\in\{1,\ldots,N\}:\varphi_i\in H^1(\Omega): -\Delta\varphi_i = 0$ *in* Ω, $\varphi_i = \delta_{ij}$ *on* Γ_j:$j = 0,1,\ldots,N$. Since $\varphi\in\dot{\varphi}\in\dot{\Theta}_2$ we always have: $\int_{\Gamma_i}\partial_\nu\varphi\, d\gamma = 0$, $i = 0,1,\ldots,N$. Hence $(d_1,\ldots,d_N)\in\mathbb{R}^N$, defined in (12), is a solution of the linear system:

$$\sum_{i=1}^{N} d_i \int_{\Gamma_j} \partial_\nu\varphi_i\, \mathrm{d}\gamma = -\int_{\Gamma_j} \partial_\nu\varphi_0\, \mathrm{d}\gamma, \quad j = 1,\ldots,N. \tag{13}$$

Note that the remaining equation on Γ_0 is then automatically verified, since φ satisfies $\int_\Gamma \partial_\nu\varphi d\gamma = 0$ and thus $\Delta\varphi\in L_0^2(\Omega)$. The linear system (13) always *possesses a unique* solution, since the columns of the matrix, associated to the linear operator, are necessarily linearly independent, and thus the rank of the associated matrix is N. ■

The set of equivalence classes, defined in (11), is easily verified to be a linear subspace of $H^1(\Omega)/\mathbb{R}$. Moreover the mapping n defined in proposition 3 is also a norm on $\dot{\Theta}_2$. To verify this, we consider an element $\dot{\varphi}\in\dot{\Theta}_2$, for which $\|\Delta\varphi\|_{0,\Omega} = 0$ for $\varphi\in\dot{\varphi}$. Consequently $\Delta\varphi = 0$ in Ω for any $\varphi\in\dot{\varphi}$. Applying now proposition 4, we have that $\varphi = c + \varphi_0 + \sum_{i=1}^N d_i\varphi_i$, and we readily deduce that $\varphi_0 \equiv 0$ in Ω. Consequently the solution $(d_1,\ldots,d_N)\in\mathbb{R}^N$ of (13) necessarily vanishes, i.e. $d_1 = \cdots = d_N = 0$ and thus $\dot{\varphi} = 0$ in $\dot{\Theta}_2$.

Concerning the mapping M defined in theorem 3, we easily verify that $M(\dot{\Theta}_1) = \dot{\Theta}_2$. We prove the following proposition:

Proposition 5. *The linear mapping* $M:(\dot{\Theta}_1, n)\to(\dot{\Theta}_2, n)$ *is continuous.*

Proof: We verify that $\dot{\psi} = M(\dot{\varphi})\in\dot{\Theta}_2$. Indeed since $\dot{\varphi}\subset H^2(\Omega)$ we have that for every $i = 0,1,\ldots,N$, $0 = \int_{\Gamma_i}\partial_\tau\varphi\, d\gamma = \int_{\Gamma_i}\partial_\nu\psi\, d\gamma$. Moreover by Green's formula, we have that $\int_\Omega \Delta\psi\, dx = \int_\Gamma \partial_\nu\varphi\, d\gamma = 0$ and hence $\Delta\psi\in L_0^2(\Omega)$. To verify that M is a continuous mapping from $\dot{\Theta}_1$ into $\dot{\Theta}_2$, we consider $\psi\in M(\dot{\varphi})$. We have for $\varphi\in\dot{\varphi}$,

$$\|\Delta\psi\|_{0,\Omega} = \|curl(\mathbf{grad}\,\varphi + \mathbf{curl}\,\psi)\|_{0,\Omega} \leqslant |\mathbf{grad}\,\varphi + \mathbf{curl}\,\psi|_{1,\Omega}.$$

Taking into account that *div* is an isomorphism, we end up with

$$|\mathbf{grad}\,\varphi + \mathbf{curl}\,\psi|_{1,\Omega} \leqslant \|div^{-1}\|_{0,\Omega}.\|\Delta\varphi\|_{0,\Omega}.$$ ■

In the sequel we will use the following notation related to the correspondence between $V^\dagger$ and $\dot{\Theta}_1$, namely $[\mathbf{grad} + \mathbf{curl}\circ M]:\dot{\varphi}\in\dot{\Theta}_1\to V^\dagger$. We will thus state for any $\mathbf{v}\in V^\dagger$ that we have $\mathbf{v} = [\mathbf{grad} + \mathbf{curl}\circ M](\dot{\varphi})$ for some $\dot{\varphi}\in\dot{\Theta}_1$, which replaces the construction of theorem 3. To conclude this section, we summarize main properties of this correspondence, by describing the basic *isomorphisms.* By choosing appropriate norms, the linear correspondence can be shown to be continuous and so does its inverse.

Theorem 4. *The mapping* $[\mathbf{grad} + \mathbf{curl}\circ M]$,

$$[\mathbf{grad} + \mathbf{curl}\circ M]:\dot{\varphi}\in(\dot{\Theta}_1, n(\cdot))\to\mathbf{v}\in(V^\dagger, |\cdot|_{1,\Omega}),$$

such that $\mathbf{v} = \mathbf{grad}\,\varphi + \mathbf{curl}\,\psi$; $\forall\,\varphi\in\dot{\varphi}$, $\psi\in M(\dot{\varphi})$, *is an isomorphism.*

Proof: Let F denote this correspondence $\dot{\varphi}\to\mathbf{v}$. This mapping is a linear bijection by virtue of theorem 3. Now, taking into account that the operator *div* is an isomorphism, we obtain that F^{-1} is a continuous mapping from $(V^{\dagger}, |\cdot|_{1,\Omega})$ onto $(\dot{\Theta}_1, n(\cdot))$. Conversely, since div^{-1} is also an isomorphism, we have that

$$|\mathbf{v}|_{1,\Omega} \leqslant \|div^{-1}\| . \|\Delta\varphi\|_{0,\Omega} = \|div^{-1}\| . n(\dot{\varphi}),$$

and thus F is also continuous from $(\dot{\Theta}_1, n(\cdot))$ onto $(V^{\dagger}, |\cdot|_{1,\Omega})$. ■

From the preceding results we can conclude that $V^{\dagger}$, $\dot{\Theta}_1$ and $L_0^2(\Omega)$ are isomorphic spaces. Moreover $(\dot{\Theta}_1, n)$ is a Hilbert space.

4. CHARACTERIZATION OF THE PRESSURE BY PRESSURE POTENTIALS

We are now ready to establish our basic result, that consists of a new *variational formulation* characterizing the pressure via associated potentials called hereafter **pressure potentials**. We further interpret this problem and show that this pressure potential class is related to the *pressure*, solution of a *Stokes problem.*

4.1. The underlying Helmholtz decomposition

The pressure P solution of a Stokes problem is contained in $L_0^2(\Omega)$, and is characterized by theorem 1. In addition we have that there exists a uniquely defined vector field $\mathbf{z}$ in $V^{\dagger}$ such that $P = div\,\mathbf{z}$. This vector field $\mathbf{z}$ is characterized in theorem 2. Moreover, based on the representation theorems, developed in section 3, we can associate a class of pressure potentials $\dot{\pi} \in \dot{\Theta}_1$, such that

$$P = div\,\mathbf{z}, \quad \mathbf{z} = [\mathbf{grad} + \mathbf{curl} \circ M](\dot{\pi}). \tag{14}$$

By (14) we mean that for any $\pi \in \dot{\pi}$ and $\zeta \in M(\dot{\pi})$ that: $\mathbf{z} = \mathbf{grad}\,\pi + \mathbf{curl}\,\zeta$, where the mapping M is defined in theorem 3.2. Obviously, we have that

$$\mathbf{f} = -\eta\,\Delta\mathbf{u} + \mathbf{grad}\,div\,\mathbf{z}, \quad in\ [H^{-1}(\Omega)]^2,$$

or with $\mathbf{u} = \mathbf{grad}\,\varphi + \mathbf{curl}\,\psi$ (for a detailed construction, see Reference 6), we obtain

$$\mathbf{f} = \mathbf{curl}(-\eta\,\Delta\psi) + \mathbf{grad}(\Delta\pi - \eta\,\Delta\varphi) \quad in\ [H^{-1}(\Omega)]^2,$$

which is a *Helmholtz decomposition* of a *weakly defined vector field* **f**.

4.2. Characterization for the pressure by a pressure potential

We formulate now our main result:

Theorem 5. *The pressure P solution of a* 2-D *Stokes problem (1) is given as* $P = \Delta\pi$, *for* $\pi \in \dot{\pi}$, $\dot{\pi}$ *being the pressure potential class, which exists and is uniquely characterized by the variational problem:*

$$?\ \dot{\pi} \in \dot{\Theta}_1, \quad (\Delta\pi, \Delta\lambda)_{0,\Omega} = \langle \mathbf{L}, [\mathbf{grad} + \mathbf{curl} \circ M]\,\dot{\lambda} \rangle, \quad \forall \dot{\lambda} \in \dot{\Theta}_1, \tag{15}$$

where $\mathbf{L}$ *is defined by* (5), $\dot{\Theta}_1$ *by* (10) *and* M *by theorem 3.2.*

Proof: By virtue of the isomorphisms described in the preceding, we have that $(\Delta., \Delta.)_{0,\Omega}$ is a bilinear continuous form on $(\dot{\Theta}_1, n(\cdot))$ which is also elliptic on this space. Similarly the linear form $\dot{\varphi} \to \langle \mathbf{L}, [\mathbf{grad} + \mathbf{curl} \circ M](\dot{\varphi}) \rangle$ is continuous on $(\dot{\Theta}_1, n(\cdot))$. We can apply the

Lax–Milgram theorem which guarantees $\dot{\pi}$, characterized by the variational problem (15), to exist and to be uniquely defined. ■

It may further be interesting to state an equivalent expression for the right-hand member of (15). We have by (5) that for $\dot{\lambda}\in\dot{\Theta}_1$:

$$\langle \mathbf{L}, [\mathbf{grad}+\mathbf{curl}\circ M](\dot{\lambda})\rangle = -\eta(curl\,\mathbf{u},\Delta\mu)_{0,\Omega} - \langle \mathbf{f}, \mathbf{grad}\,\lambda + \mathbf{curl}\,\mu\rangle_{[H_0^1(\Omega)]^2},$$

where $\mu\in M(\dot{\lambda})$.

In the **particular case** (of considerable practical importance) where the domain is *simply connected* and *sufficiently regular*, such that for smoother data as $\mathbf{f}\in H(curl;\Omega)$ and $\mathbf{g}\in[H^{+3/2}(\Gamma)]^2$ we have that $\Delta\mathbf{u}\in[L^2(\Omega)]^2$, then further simplifications can be obtained for the right-hand member of (15). Indeed it is readily verified that:

$$\langle \mathbf{L}, [\mathbf{grad}+\mathbf{curl}\circ M](\dot{\lambda})\rangle = -\eta\int_\Gamma curl\,\mathbf{u}\,\partial_\tau\lambda\,\mathrm{d}\gamma - (\mathbf{f},\mathbf{grad}\,\lambda)_{0,\Omega}, \quad \forall\,\dot{\lambda}\in\dot{\Theta}_1,$$

which offers the advantage of avoiding the *complementary construction* of $\dot{\mu}=M(\dot{\lambda})$ for any $\dot{\lambda}$ in $\dot{\Theta}_1$ as required by the preceding. In these cases the Stokes problem amounts to determining some stream function and some pressure potential from the respective equivalence classes. What is important to remark at this stage is that the determination of a pressure potential can be performed by the same procedure as those of the stream function without major modifications. Indeed

$$? \ \psi\in H^2(\Omega):\partial_\nu\psi = -\mathbf{g}\cdot\boldsymbol{\tau} \ on\ \Gamma, \quad \psi = H \ on\ \Gamma, \quad \psi=0 \ at\ x_0\in\Gamma$$

$$where \quad H\in H^{+3/2}(\Gamma):DH=\mathbf{g}\cdot\mathbf{v},\ on\ \Gamma$$

$$(\Delta\psi,\Delta\chi)_{0,\Omega} = \frac{1}{\eta}(\mathbf{f},\mathbf{curl}\,\chi)_{0,\Omega}, \quad \forall\,\chi\in H_0^2(\Omega),$$

$$? \ \dot{\pi}\in H^2(\Omega):\partial_\nu\pi = 0 \ on\ \Gamma, \quad \pi=0 \ at\ x_0\in\Gamma$$

$$(\Delta\pi,\Delta\lambda)_{0,\Omega} = -(\mathbf{f},\mathbf{grad}\,\lambda)_{0,\Omega} + \eta\int_\Gamma \Delta\psi\,\partial_\tau\lambda\,\mathrm{d}\gamma,$$

$$\forall\,\dot{\lambda}\in H^2(\Omega):\partial_\nu\lambda=0 \ on\ \Gamma, \quad \lambda=0 \ at\ x_0\in\Gamma.$$

The main questions to be answered now are as follows. *Can the variational problem (15) be interpreted as a Stokes problem and are there some boundary conditions which are implicitly verified?* Before proceeding further, we want to point out that for the pressure potential class $\dot{\pi}$, *to be linked up with the pressure, we also have, by virtue of (1), that,* $\Delta P = div\,\mathbf{f}$ *in* Ω, and thus for every $\pi\in\dot{\pi}$,

$$\Delta^2\pi = div\,\mathbf{f} \quad in\ \Omega. \tag{16}$$

Consequently from the interpretation of the problem, solved by theorem 5, we are expecting (15) to be equivalent with some biharmonic problem (16), possibly together with 'appropriate boundary conditions' to be detailed hereafter.

4.3. *Interpretation of the problem solved by (15)*

We first remark that the equivalence classes $\dot{\chi}$, defined as

$$\dot{\chi} = \{\chi + c\,|\,\chi\in D(\Omega), c\in\mathbb{R}\},$$

are elements of $\dot{\Theta}_1$, where $D(\Omega)$ represents the space of test functions. Let us consider the class $\dot{\pi}\in\dot{\Theta}_1$ solution of (15), which by virtue of theorem 5 exists and is uniquely characterized. Now $[\mathbf{grad}+\mathbf{curl}\circ M](\dot{\chi})=\mathbf{grad}\,\chi$ for $\chi\in\dot{\chi}$, and is an immediate consequence from the definition of M. Moreover $\mathbf{grad}\,\chi\in[H_0^1(\Omega)]^2$ and thus

$$\langle \mathbf{f}, [\mathbf{grad}+\mathbf{curl}\circ M](\dot{\chi})\rangle_{[H_0^1(\Omega)]^2} = -\langle div\,\mathbf{f}, \chi\rangle_{[H_0^1(\Omega)]^2}. \tag{17}$$

The second term of the right-hand member is easily verified to reduce to zero. Finally the left-hand member of (15) becomes

$$(\Delta\pi, \Delta\chi)_{0,\Omega} = -\langle \mathbf{grad}\,\pi, \mathbf{grad}\,\chi\rangle_{[H_0^1(\Omega)]^2} = \langle \Delta^2\pi, \chi\rangle.$$

Taking now into account (17) and the preceding, we deduce from (15) that

$$\langle \Delta^2\pi - div\,\mathbf{f}, \chi\rangle = 0, \quad \forall\,\chi\in D(\Omega).$$

The latter equation proves that the distributions contained in $\dot{\pi}\in\dot{\Theta}_1$ satisfy (16) in $D(\Omega)^*$.

4.4. *Interpretation of the boundary conditions realized by (15)*

A most interesting and intricate topic to be discussed now consists in discovering which boundary conditions are *implicitly* realized by solving the variational problem (15). Indeed, the *natural boundary conditions* which are realized, besides the *essential boundary conditions* $\partial_\nu\pi=0$ on Γ (and resulting from the construction of $\dot{\pi}$ in $\dot{\Theta}_1$), are **unknown up to now**. To verify which natural boundary conditions are verified, we use the technique (Reference 8, p. 20, 25), which mainly consists in assuming additional smoothness of the solution.

Let $\dot{\pi}\in\dot{\Theta}_1$ satisfy $\dot{\pi}\subset H^4(\Omega)$. By repeated use of Green's formula, we obtain for any $\dot{\lambda}\in\dot{\Theta}_1$ that for $\pi\in\dot{\pi}$ we have

$$(\Delta\pi, \Delta\lambda)_{0,\Omega} = (\Delta^2\pi, \lambda)_{0,\Omega} - \int_\Gamma \partial_\nu\Delta\pi\,\lambda\,d\gamma + \int_\Gamma \partial_\tau\Delta\pi\,\mu\,d\gamma, \tag{18}$$

where $\mu\in M(\dot{\lambda})$. Similarly, under the present regularity, which amounts to requiring $\mathbf{f}\in H(div;\Omega)\cap H(curl;\Omega)$, the first term of the right-hand side of (15) can be stated as:

$$\langle \mathbf{f}, [\mathbf{grad}+\mathbf{curl}\circ M](\dot{\lambda})\rangle = -(div\,\mathbf{f}, \lambda)_{0,\Omega} + (curl\,\mathbf{f}, \mu)_{0,\Omega} + \int_\Gamma \mathbf{f}\cdot\mathbf{v}\,\lambda\,d\gamma - \int_\Gamma \mathbf{f}\cdot\boldsymbol{\tau}\,\mu\,d\gamma. \tag{19}$$

The second term of the right-hand member of (15) is easily verified to satisfy

$$a(\mathbf{u}, [\mathbf{grad}+\mathbf{curl}\circ M](\dot{\chi})) = -\eta[(\Delta\mathbf{u}, \mathbf{grad}\,\lambda)_{0,\Omega} + (\Delta\mathbf{u}, \mathbf{curl}\,\mu)_{0,\Omega}], \tag{20}$$

for any $\mathbf{u}\in H^2(\Omega)$, satisfying $div\,\mathbf{u}=0$ and thus also for the velocity field solution of the Stokes problem. We note that (19) and (20) are not altered, by taking $\lambda+c$ or $\mu+d$, for any $c,d\in\mathbb{R}$ rather than some $(\lambda,\mu)\in\dot{\lambda}\times M(\dot{\lambda})$. Otherwise stated, we have

$$a(\mathbf{u}, [\mathbf{grad}+\mathbf{curl}\circ M](\dot{\lambda})) = -\eta\left[(curl\,\Delta\mathbf{u}, \mu)_{0,\Omega} + \int_\Gamma (\Delta\mathbf{u})\cdot\mathbf{v}\,\lambda\,d\gamma - \int_\Gamma (\Delta\mathbf{u})\cdot\boldsymbol{\tau}\,\mu\,d\gamma\right]. \tag{21}$$

In the light of (16) and taking into account (18), (19) and (21) we conclude that (15) reduces to

$$\int_\Gamma [\mathbf{f}\cdot\boldsymbol{\tau} - \partial_\tau(\Delta\pi) + \eta(\Delta\mathbf{u})\cdot\boldsymbol{\tau}]\,\mu\,d\gamma - \int_\Gamma [\mathbf{f}\cdot\mathbf{v} - \partial_\nu(\Delta\pi) + \eta(\Delta\mathbf{u})\cdot\mathbf{v}]\,\lambda\,d\gamma = 0,$$

$$\forall \dot{\lambda} \in \dot{\Theta}_1 \quad \text{and} \quad \mu \in M(\dot{\lambda}). \tag{22}$$

Let us denote for short, $\mathbb{F} = \mathbf{f} - \mathbf{grad}\,\Delta\pi + \eta\,\Delta\mathbf{u}$. We then note that we already have that:

$$div\,\mathbb{F} = 0 \; in \; \Omega, \quad curl\,\mathbb{F} = 0 \; in \; \Omega, \tag{23}$$

by virtue of (16). From (23) we further deduce:

$$\int_\Gamma \mathbb{F}\cdot\mathbf{v}\,d\gamma = 0 \quad \text{and} \quad \int_\Gamma \mathbb{F}\cdot\boldsymbol{\tau}\,d\gamma = 0. \tag{24}$$

Moreover, the resulting equation (22) can now be stated as:

$$\int_\Gamma [\mathbb{F}\cdot\mathbf{v}\,\lambda - \mathbb{F}\cdot\boldsymbol{\tau}\,\mu]\,d\gamma = 0, \quad \forall \dot{\lambda} \in \dot{\Theta}_1, \quad \mu \in M_1(\dot{\lambda}). \tag{25}$$

Taking then into account that for any $\lambda^* \in \Psi$ we have that $\dot{\lambda} \doteq \{\lambda^* + c \mid c \in \mathbb{R}\}$ is contained in $\dot{\Theta}_1$, we obtain thus that

$$\int_{\Gamma_i} \mathbb{F}\cdot\mathbf{v}\,d\gamma = 0, \quad i = 0,\ldots,N. \tag{26}$$

Consequently we have for $\mathbb{F} \in [L^2(\Omega)]^2$ that (23) and (26) are valid, and hence we conclude by virtue of Reference 9 that there *exists a unique stream function class*, which represents $\mathbb{F}$, i.e. there exists a unique $\dot{\vartheta} \in H^1(\Omega)/\mathbb{R}$ such that $\mathbb{F} = \mathbf{curl}\,\vartheta$ *in* $[L^2(\Omega)]^2$. Since $\mathbb{F} \in H(div;\Omega) \cap H(curl;\Omega)$ (cfr. (23)), we have that the associated stream function class $\dot{\vartheta}$ is a solution of the following Neumann problem:

$$\dot{\vartheta} \in H^1(\Omega)/\mathbb{R}: -\Delta\vartheta = 0 \; in \; \Omega, \quad \partial_\nu\vartheta = -\mathbb{F}\cdot\boldsymbol{\tau} \; on \; \Gamma, \tag{27}$$

the compatibility condition being verified by virtue of (24). Consequently (25) now becomes

$$\int_\Gamma [\lambda\,\partial_\tau\vartheta + \mu\,\partial_\nu\vartheta]\,d\gamma = 0, \quad \forall \dot{\lambda} \in \dot{\Theta}_1,\ \mu \in M(\dot{\lambda}), \tag{28}$$

We are now going to show that (28) implies $\vartheta \equiv 0$ in Ω and thus that (25) implies $\mathbb{F}$ to be a *zero vector field* in Ω, completing this way the proof that the determination of the pressure potential according to (15) solves the Stokes problem for $P = \Delta\pi$. For a simply connected domain the proof is surprisingly simple, and therefore we distinguish hereafter two cases.

If the domain Ω is **simply connected,** then the relation (28) reduces to $\int_\Gamma \vartheta\,\partial_\tau\lambda\,d\gamma = 0$ for $\forall \dot{\lambda} \in \dot{\Theta}_1$, by virtue of (24). Since now $\dot{\lambda} \in \dot{\Theta}_1$ implies that $\partial_\tau\lambda \in [H^{-1/2}(\Gamma)]$ (Reference 1, p. 5) and by Reference 7, p. 165, we obtain that $\vartheta = 0$ on Γ in $H^{+1/2}(\Gamma)$. Taking now into account that $\vartheta \in H^1(\Omega)$ satisfies $\Delta\vartheta = 0$ in Ω and $\vartheta = 0$ on Γ, we conclude that $\vartheta \equiv 0$ in $\bar{\Omega}$ and thus that $\mathbb{F} \equiv 0$ in $\bar{\Omega}$.

In cases where the domain Ω is **multiply-connected** we will interpret (28) further and explicitly state the functional relation between λ and the trace of some associated μ. With some class $\dot{\lambda} \in \dot{\Theta}_1$, there is associated a class $\dot{\mu} \in \dot{\Theta}_2$ characterized as follows: $\dot{\mu} = \{\mu + c \mid c \in \mathbb{R}\}$ where

$$\mu = \mu^* + \mu_0 + \sum_{i=1}^{N} c_i \psi_i,$$

each term being defined by the problems

(i) $\mu^* \in H^2(\Omega): \mu^* = 0$ *on* Γ, $\partial_\nu \mu^* = \partial_\tau \lambda$ *on* Γ,
(ii) $\mu_0 \in H_0^2(\Omega): (\Delta\mu_0, \Delta\chi)_{0,\Omega} = -(\Delta\mu^*, \Delta\chi)_{0\Omega}, \quad \forall \chi \in H_0^2(\Omega)$,
(iii) ψ_i for $i = 1, \dots, N$ defined by (8),
(iv) $(c_1, \dots, c_N) \in \mathbb{R}^N$

$$\sum_{i=1}^{N} c_i(\Delta\psi_i, \Delta\psi_j)_{0,\Omega} = -(\Delta\mu^*, \Delta\psi_j)_{0,\Omega}, \quad j = 1, \dots, N. \tag{29}$$

It is important to stress that the linear system (29) characterizing $(c_1, \dots, c_N)$ possesses a unique solution. Indeed the associated linear operator is positive definite, since $\|\Delta.\|_{0,\Omega}$ is a norm on Ψ. We are particularly interested in the traces of $\mu \in M(\lambda)$, on $\Gamma_0, \dots, \Gamma_N$. Now by construction we have that: $\mu = 0$ *on* Γ_0, $\mu = c_i$ *on* $\Gamma_i; i = 1, \dots, N$, where $\mathbf{c} = (c_1, \dots, c_N)$ is characterized by (29) or by the equivalent linear system

$$\sum_{i=1}^{N} c_i(\Delta\psi_i, \Delta\psi_j)_{0,\Omega} = -\int_\Gamma \Delta\psi_j \partial_\tau \lambda \, d\gamma \quad j = 1, \dots, N,$$

shortly noted: $L(\mathbf{c}) = I(\partial_\tau \lambda)$, where we introduce the map

$$I: \omega \in H^{+1/2}(\Gamma) \to \left(-\int_\Gamma \Delta\psi_j \omega \, d\gamma \,\middle|\, j = 1, \dots, N \right) \in \mathbb{R}^N.$$

We can replace now (28) by

$$\int_\Gamma [\vartheta - \partial_\nu \vartheta L^{-1} \circ I] \partial_\tau \lambda \, d\gamma = 0, \quad \forall \lambda \in \dot{\Theta}_1,$$

from which we deduce that $\vartheta = \partial_\nu \vartheta \; L^{-1} \circ I$ *on* Γ, which can only be realized provided $\vartheta = 0$ and $\partial_\nu \vartheta = 0$ *on* Γ. Consequently, by virtue of the characterization (27), we *conclude* that $\mathbb{F} \equiv 0$ in $\bar{\Omega}$.

Finally, *we summarize that by solving (15) we are formally solving the problem:*

$$?\dot{\pi}: \Delta^2 \pi = \mathit{div}\, \mathbf{f} \quad \mathit{in} \quad \Omega,$$

$$\partial_\nu \pi = 0 \quad \mathit{on} \quad \Gamma,$$

$$\mathbf{grad}\, \Delta\pi = \mathbf{f} + \eta \Delta\boldsymbol{\mu}, \quad \mathit{in} \quad \bar{\Omega}.$$

We then conclude, by virtue of the preceding, that (15) determines the pressure potential class $\dot{\pi}$ and that the function P, defined as $P = \Delta\pi$, $\pi \in \dot{\pi}$, satisfies $\mathbf{grad}\, P = \mathbf{f} + \eta \Delta \mathbf{u}$, *in* $\bar{\Omega}$, which formally corresponds with the pressure, characterized by a 2-*d* Stokes problem. Indeed we have on the one hand:

$$\mathbf{grad}\, P = \mathbf{f} + \eta \Delta \mathbf{u}, \quad \mathit{in} \quad \Omega,$$

and on the other hand, with $\omega = \mathit{curl}\, \mathbf{u}$:

$$\partial_\tau P = \mathbf{f} \cdot \boldsymbol{\tau} + \eta \partial_\nu \omega, \quad \mathit{on} \quad \Gamma$$

$$\partial_\nu P = \mathbf{f} \cdot \mathbf{v} - \eta \partial_\tau \omega, \quad \mathit{on} \quad \Gamma$$

Consequently, under the present assumptions of regularity on the data we have that the boundary conditions, *implicitly realized* by the variational characterization of the pressure potential, are both of a Dirichlet and Neumann type on the *pressure*.

5. FINITE ELEMENT APPROXIMATIONS

A finite element approximation of the problem (15) will be obtained by considering a finite element subspace $\dot{V}_h \subset H^1(\Omega)/\mathbb{R}$, where every element in the equivalence class is composed of piecewise polynomials from $C^1(\bar{\Omega})$. Consequently these functions are constructed by, for example, *Argyris triangles*. The problem becomes

$$? \ \dot{\pi}_h \in \dot{V}_h \cap \dot{\Theta}_1, \quad (\Delta\pi_h, \Delta\lambda)_{0,\Omega} = L_h(\dot{\lambda}), \quad \forall\, \dot{\lambda} \in \dot{V}_h \cap \dot{\Theta}_1, \tag{30}$$

where $L_h(\dot{\lambda}) = -\eta(curl\,\mathbf{u}_h, \Delta\mu)_{0,\Omega} - \langle\, \mathbf{f}, \mathbf{grad}\,\lambda + \mathbf{curl}\,\mu \,\rangle_{[H_0^1(\Omega)]^n}$, and $\dot{\mu} \in \dot{V}_h \cap \dot{\Theta}_2$ corresponds with $\dot{\lambda}$ according to:

$$\partial_\tau \mu = 0 \quad on\ \Gamma, \quad \partial_\nu \mu = \partial_\tau \lambda \quad on\ \Gamma, \quad (\Delta\mu, \Delta\chi)_{0,\Omega} = 0, \quad \forall\, \chi \in V_h \cap \Psi.$$

In the latter construction of $\dot{\mu}$, it is essential to realize exactly the matching at the boundary Γ, between λ and μ, such that **grad** λ + **curl** $\mu \in [H_0^1(\Omega)]^2$. Indeed this is an *essential* boundary condition, which stems from the mixed variational principle, and is thus to be exactly realized on the whole of Γ. It follows that the *Bell triangle* (even with $V_h \subset C^1(\bar{\Omega}) \cap H^2(\Omega)$) is **not suited** for this purpose, since on Γ the tangential derivative is a polynomial of degree 4, whereas the normal derivative is by construction a polynomial of degree 3. That this problem (30) is well-posed and possesses a unique solution follows from the properties of the forms involved, as shown in the preceding.

Regarding the *convergence properties*, we readily deduce from the *first Strang lemma* (Reference 8, p. 186) that there exists a constant $C > 0$ independent of the subspace $\dot{V}_h$, such that

$$\|\Delta\pi - \Delta\pi_h\|_{0,\Omega} \leqslant C\Bigg[inf\{\, \|\Delta\pi - \Delta\lambda\|_{0,\Omega} | \dot{\lambda} \in \dot{V}_h \cap \dot{\Theta}_1 \} + \eta\, sup\left\{ \frac{|(\Delta\psi - \Delta\psi_h, \Delta\mu)_{0,\Omega}|}{\|\Delta\lambda\|_{0,\Omega}} \Big| \dot{\lambda} \in \dot{V}_h \cap \dot{\Theta}_1 \right\} \Bigg].$$

Consequently by introducing the interpolation operator Π_h, related to the finite dimensional space V_h, we have

$$inf\{\, \|\Delta\pi - \Delta\lambda\|_{0,\Omega} | \dot{\lambda} \in \dot{V}_h \cap \dot{\Theta}_1 \} = d(\dot{\pi}, \dot{V}_h \cap \dot{\Theta}_1) \leqslant \|\Delta\pi - \Delta\Pi_h(\pi)\|_{0,\Omega}.$$

For every $\lambda \in \dot{\lambda}$, $\dot{\lambda} \in \dot{\Theta}_1$, we have

$$\|\Delta\lambda\|_{0,\Omega} \leqslant |[\mathbf{grad} + \mathbf{curl} \circ M](\dot{\lambda})|_{1,\Omega} \leqslant |\lambda|_{2,\Omega} + |\mu|_{2,\Omega}, \quad \mu \in M(\dot{\lambda}).$$

We note now that the operators M and Π_h *commute* (i.e. $M \circ \Pi_h = \Pi_h \circ M$), since the matching at the boundary between λ and μ is supposed to be exactly realized over the whole of Γ. We have then:

$$d(\dot{\pi}, \dot{V}_h \cap \dot{\Theta}_1) \leqslant C_1[|\pi - \Pi_h(\pi)|_{2,\Omega} + |\zeta - \Pi_h(\zeta)|_{2,\Omega}], \quad \zeta \in M(\dot{\pi}),$$

On the other hand:

$$|(\Delta\psi - \Delta\psi_h, \Delta\mu)_{0,\Omega}| \leqslant C_2 \|\psi - \psi_h\|_{2,\Omega} \cdot \|\Delta\lambda\|_{0,\Omega},$$

by taking into account that: $\|\Delta\psi - \Delta\psi_h\|_{0,\Omega} \leqslant C_3 |\psi - \psi_h|_{2,\Omega}$, and by virtue of proposition 5, that: $\|\Delta\mu\|_{0,\Omega} \leqslant C_4 \|\Delta\lambda\|_{0,\Omega}$. Consequently we end up with the following appraisal:

$$sup\left\{ \frac{|(\Delta\psi - \Delta\psi_h, \Delta\mu)_{0,\Omega}|}{\|\Delta\lambda\|_{0,\Omega}} \Big| \dot{\lambda} \in \dot{V}_h \cap \dot{\Theta}_1 \right\} \leqslant C_2 \|\psi - \psi_h\|_{2,\Omega}.$$

Finally, we deduce by using classical convergence results (Reference 8, p. 355) that

$$\|P - P_h\|_{0,\Omega} \subset O(h^4),$$

In cases where both the stream function and the space V_h are constructed by using *Argyris triangles*, and under the usual hypotheses that the data (and in particular the stream function) and the pressure are sufficiently regular. The preceding methods, by determining the pressure via the use of *pressure potentials*, lead to numerical methods that of course are more elaborate than the mixed variational method for the Stokes problem, but provide the advantage of offering faster convergent numerical algorithms.

6. SOME NUMERICAL EXAMPLES

In Figures 1 to 3, we consider some practical situations. In each case, we consider a Stokes problem in some cavity. The boundary conditions on the velocity are: a parabolic in- (and out-) flow profile, and no slip conditions on the upper and lower wall. If $2u_m$ is the total flux of the fluid per unit depth, we represent

$$\frac{\psi}{u_m}, \quad \frac{\pi}{\eta u_m},$$

which are the corresponding unnormalized stream function (Figures 1(a), 2(a), 3(a)) and unnormalized pressure potential (Figures 1(b), 2(b), 3(b)). The latter pressure potential is an element of $\dot{\pi} \in \dot{\Theta}_1$, and is represented by the drawing of equipotential lines.

7. CONCLUSIONS

We have presented a new theoretical approach to the pressure characterization in Stokes problems ($\mathbb{R}^n$). In particular, for the 2-dimensional problem, it is demonstrated that the Stokes problem can indeed be decoupled first, as is well known, into an independent problem for the velocity field by using a stream function, and secondly into a newly defined problem, characterizing the *pressure* via a *potential* (*-class* $\dot{\pi}$), called here the *pressure potential* (*-class*), such that $P = \Delta\pi$. The latter problem requires a knowledge of the velocity field, as is intuitively expected. Such a velocity field can easily be obtained by using a stream function formulation. The main result consists in the variational formulation (15) which characterizes uniquely the pressure potential class $\dot{\pi}$ (in $\dot{\Theta}_1$). The main interest of these results, remains in the well-posedness of the problem, which moreover uniquely characterizes this pressure potential class. The most intricate question about this pressure characterization is the interpretation of the variational problem. As is shown in the section 4.4, the problem solved by the variational equations can exactly be interpreted as the Stokes problem, which implicitly realizes the equation of motion *without stating explicitly, any boundary condition on the pressure.* These new variational equations can easily be realized by finite element approximations. As was pointed out in section 5, these problems are well posed. Moreover, pressure potentials realized by Argyris triangles offer the advantage of ensuring an $O(h^4)$ convergence for the $\|\cdot\|_{0,\Omega}$ norm of the pressure.

These results on the pressure determination, reported in section 4, rely on an interpretation of the operator formulation of the Stokes problem. From this analysis, we deduce two basic characterizations for the pressure described by a Stokes problem in $\mathbb{R}^n$ (section 2). However these results are of less numerical interest, since the construction of finite element subspaces of $V^\dagger$ is obviously very difficult to realize. That our results from section 2 could finally be transformed into

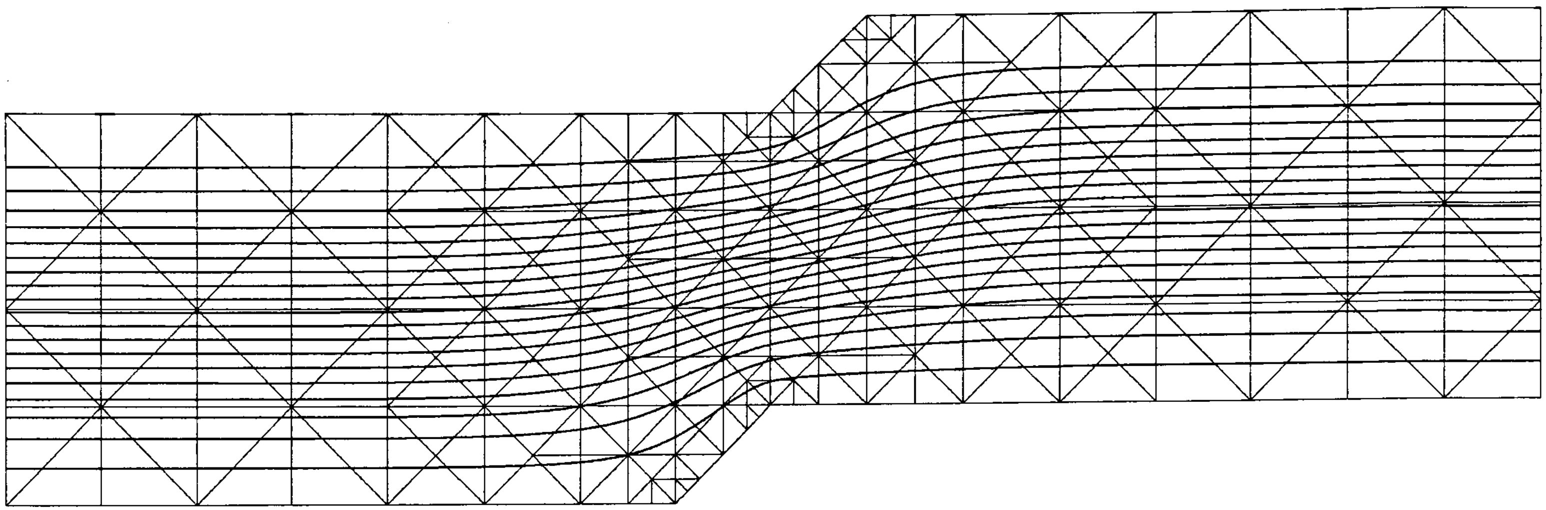

Figure 1(a). Stream functions

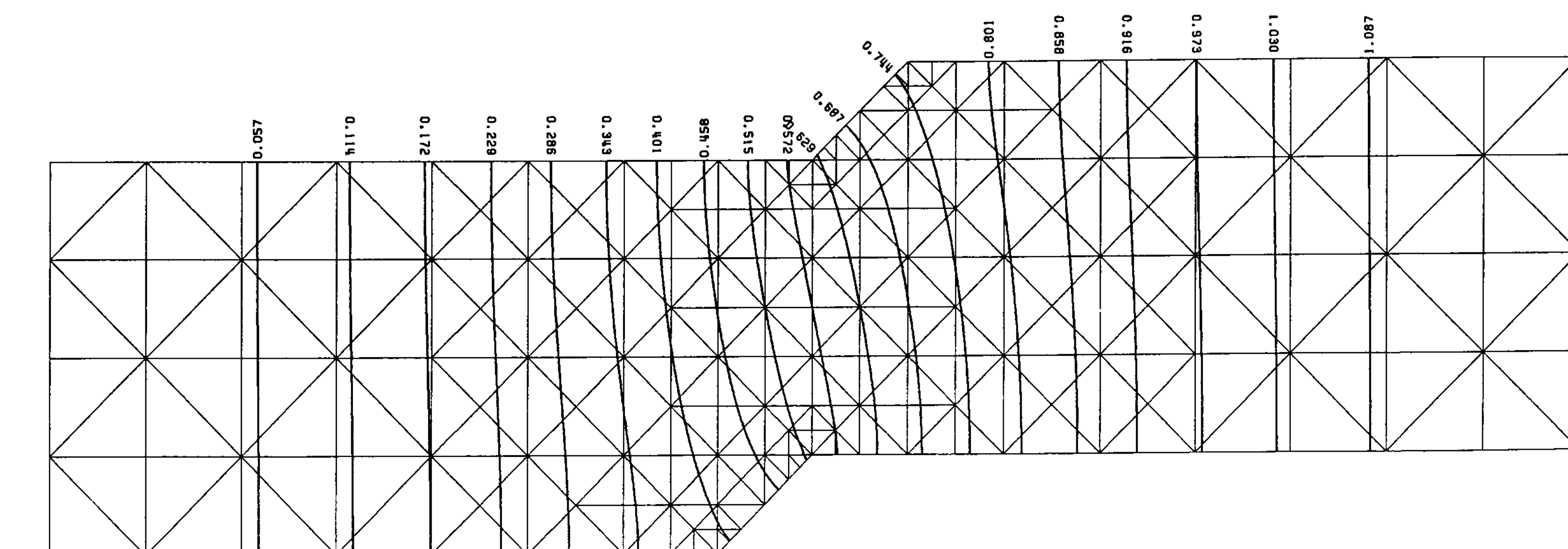

Figure 1(b). Pressure potential

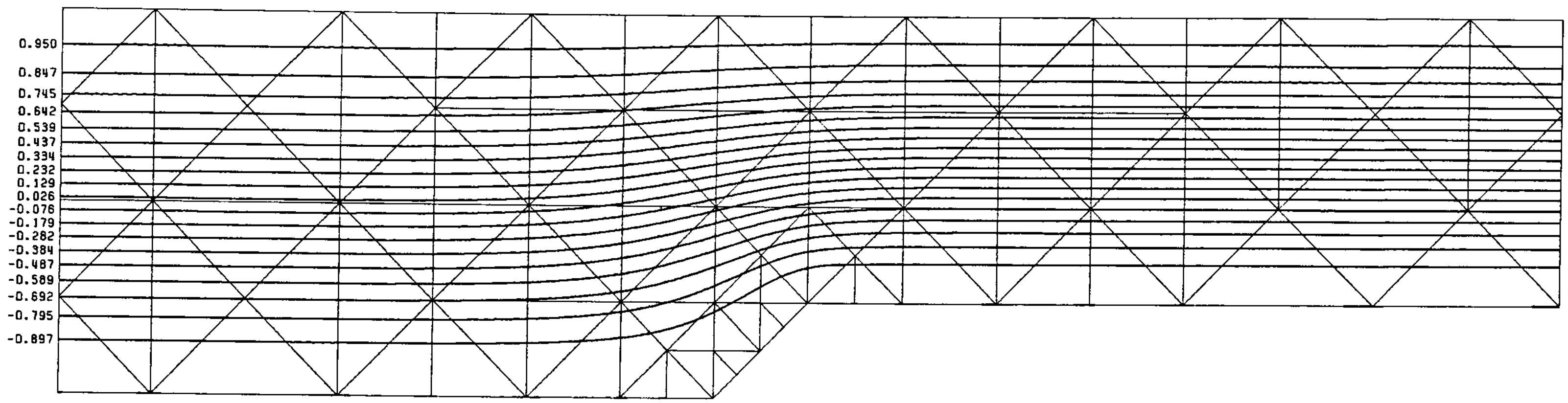

Figure 2(a). Stream functions

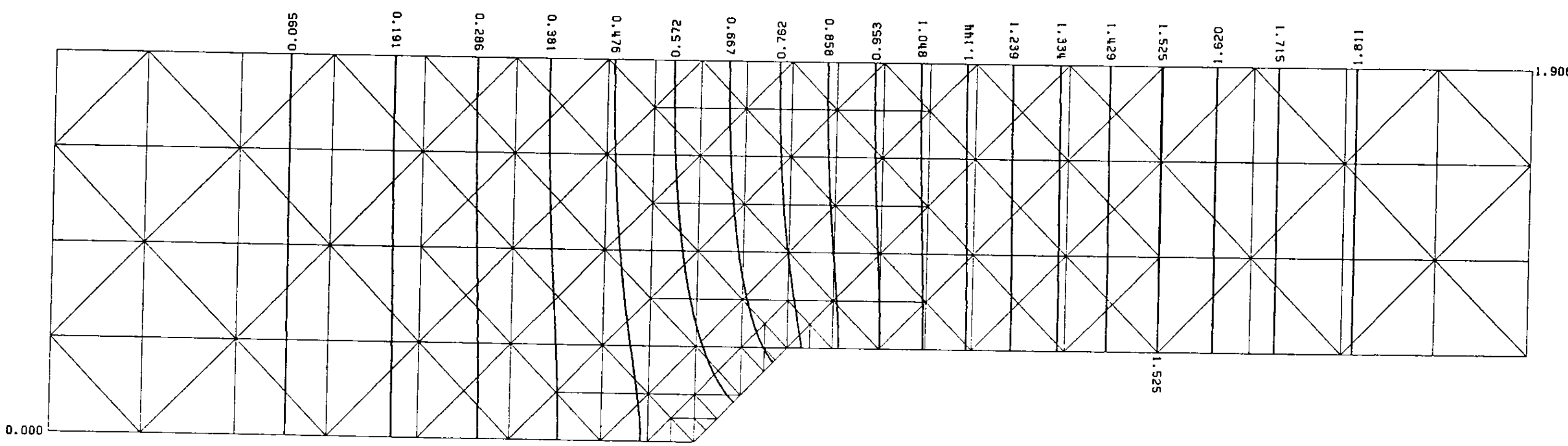

Figure 2(b). Pressure potential

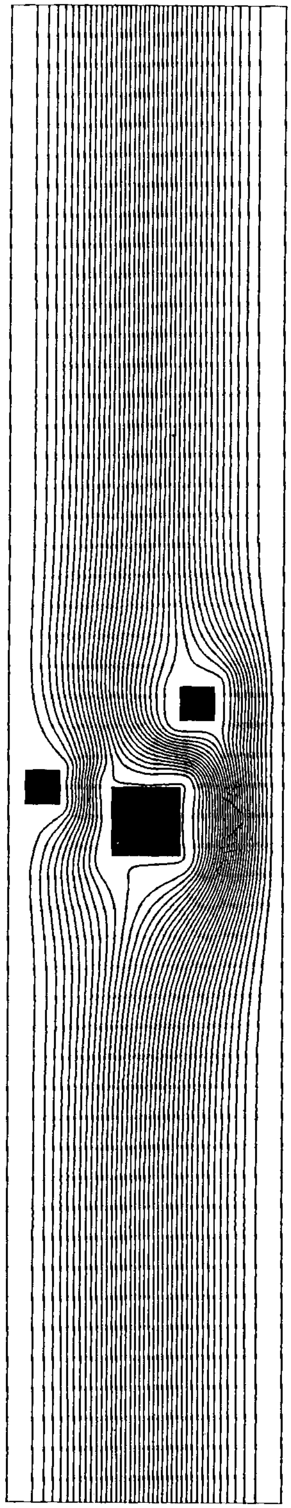

Figure 3(a). Stream function

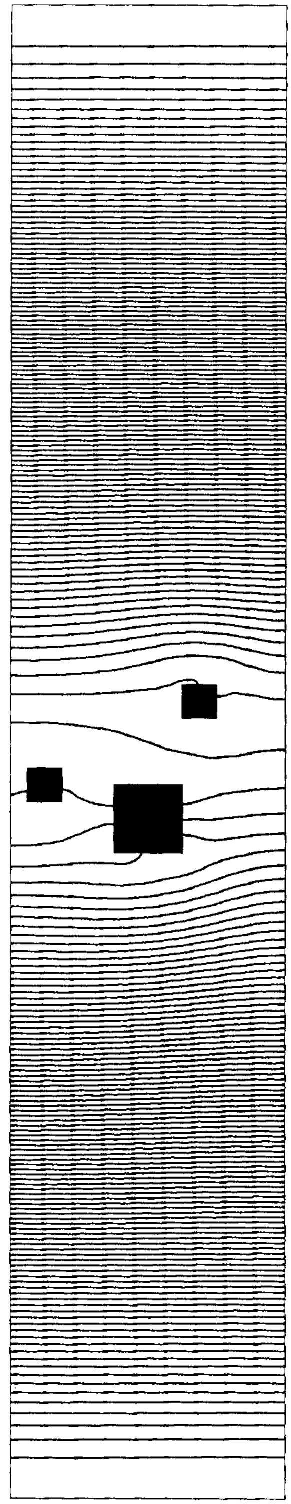

Figure 3(b). Pressure potential

variational principles of practical value was made possible on the one hand by restriction on to the *two-dimensional* case, and on the other hand by the analysis and results of section 3 about the *representation* by *potentials*, of vector fields in $V^{\dagger}$.

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